THE EFFECT OF ADDITIONAL WEIGHT COMPONENTS ON THE OPTIMIZATION OF MOTION FOR LIMITED POWER

(DOPOLNITEL'NYE VESOVYE KOMPONENTY V ZADACHAKH OPTIMIZATSII DVIZHENIIA S OGRANICHENNOI MOSHCHNOST'IU)

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Iu.N. IVANOV
(Moscow)

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In the majority of papers on the optimum regime of motion for limited power it is assumed that the vehicle consists of three parts: payload, the weight of the power source, and the weight of the working substance [1]. A more detailed analysis requires the inclusion of additional weight components in the weight formula.

The qualitative features of optimum guidance are investigated here, taking into account: (1) the weight of the motor, and (2) the weight of the reactive mass of the power source.

We introduce the following notation: q, V, P and N are the mass flow, exhaust velocity, thrust and power of the exhaust jet, respectively; G_m, G_N and G_n are the weight of the propellant, the weight of the power source, and the payload, respectively; G_p and G_ε are the weights of the motor and reactive mass of the power source, respectively; G is the total weight, $a = Pg/G$ is the acceleration due to the thrust, $\mathbf{r}$ and $\mathbf{v}$ are the radius vector and velocity vector of the moving point, $\mathbf{i}$ is a unit vector in the direction of the thrust, $\mathbf{R}(\mathbf{r}, t)$ is the acceleration of gravity, and t is the time – the argument of the problem. We have the following relations between the weight components G_m and G_N and the parameters of the exhaust jet:

$$G_N = \alpha N_{\max}, \qquad G^{\cdot}_m = -gq = -\frac{P^2 g}{2N} = -\frac{a^2 G^2}{2gN} \tag{0.1}$$

The motion of the point obeys the following system of equations and boundary conditions:

$$\mathbf{r}^{\cdot} = \mathbf{v}, \qquad \mathbf{v}^{\cdot} = (Pg / G)\,\mathbf{i} + \mathbf{R} = a\mathbf{i} + \mathbf{R} \tag{0.2}$$
$$\mathbf{r}(0) = \mathbf{r}^{(0)}, \qquad \mathbf{v}(0) = \mathbf{v}^{(0)}, \qquad \mathbf{r}(T) = \mathbf{r}^{(1)}, \qquad \mathbf{v}(T) = \mathbf{v}^{(1)}$$

where T is the time of motion and the superscripts 0 and 1 refer to the beginning and the end of the motion.

The optimum control functions of the problem are selected on the following basis: It is required to provide the maximum payload G_n for fixed initial weight $G^{(0)}$ and prescribed initial $(\mathbf{r}^{(0)}, \mathbf{v}^{(0)})$ and final $(\mathbf{r}^{(1)}, \mathbf{v}^{(1)})$ points in phase space. The time of motion T is also given.

1. We formulate the variational problem for an ideal propulsion system, taking into account the weight of the motor

$$G = G_n + G_m + G_N + G_P \tag{1.1}$$

The weight of the motor G_P is a function of the maximum power N_{max} and the maximum thrust P_{max} delivered by the motor, as well as a number of parameters $b_1, \ldots, b_s$

$$G_P = f(N_{max}, P_{max}, b_1, \ldots, b_s) \tag{1.2}$$

The parameters $b_1, \ldots, b_s$ for an ideal propulsion system do not enter into the dynamical equations (0.2), and they appear in the weight formula (1.1) only through the term (1.2), hence they may be chosen from the condition of minimum G_P before the whole variational problem has been solved. The resulting expressions take the form

$$b_1 = \varphi_1(N_{max}, P_{max}), \ldots, b_s = \varphi_s(N_{max}, P_{max}) \tag{1.3}$$

Substitution of (1.3) into (1.2) yields the functional dependence

$$G_P = F(N_{max}, P_{max}) \tag{1.4}$$

The parameters N_{max} and P_{max} are to be found from the solution of the whole variational problem. We will describe the subsequent procedure for the case where (1.4) is the linear function

$$G_P = \alpha' N_{max} + \gamma P_{max} \tag{1.5}$$

The order of magnitude of γ is $10^3 - 10^5$ [2].

In view of the fact that the weights G_P and G_N enter additively into Formula (1.1) for the total weight, the term $\alpha' N_{max}$ from (1.5) may be added to G_N $(G_N + \alpha' N_{max} = (\alpha + \alpha') N_{max})$. In this manner the component of the motor weight in (1.5) proportional to N_{max} may be excluded from consideration. For the weight of the motor we will use Formula

$$G_P = \gamma P_{max} \tag{1.6}$$

We refer the thrust P to the maximum thrust $P_{max} = G_P / \gamma$, the power N to the maximum power $N_{max} = G_N / \alpha$ and all of the weights G_n, G_m, G_N and G_P to the initial weight $G^{(0)}$ retaining the old notation for the new dimensionless quantities. As in [3] we introduce the weight

$$G_\Sigma = G_n + G_m \qquad (G_\Sigma^{(1)} = G_n) \tag{1.7}$$

Then the system of equations for the rate of efflux and for the dynamics is written

$$G_\Sigma^{\cdot} = -\frac{\alpha g}{2\gamma^2} \frac{G_P^2 P^2}{G_N N}, \qquad \mathbf{r}^{\cdot} = \mathbf{v}, \qquad \mathbf{v}^{\cdot} = \frac{g}{\gamma} \frac{P G_P}{G_\Sigma + G_N + G_P} \mathbf{i} + \mathbf{R} \tag{1.8}$$

The initial condition for the weight G_Σ takes the form

$$G_\Sigma^{(0)} = 1 - G_P^{(0)} - G_N^{(0)} \tag{1.9}$$

In accordance with the variational problem formulated above, it is necessary to find the optimum control functions $\mathbf{i}$, N, P, G_N, and G_P, giving the maximum relative payload $G_n = G_\Sigma^{(1)}$, for which the motion obeys the differential equations (1.8) and the boundary conditions (1.1) and (1.9).

By definition the control functions $N(t)$ and $P(t)$ are bounded above and below

$$1 \geqslant N(t) \geqslant 0, \qquad 1 \geqslant P(t) \geqslant 0$$

The control functions $G_N(t)$ and $G_P(t)$ differ from the others in that their initial values $G_N^{(0)}$ and $G_P^{(0)}$ are related by condition (1.9). In order to apply the extremum method of variational analysis, we supplement the system (1.8) by the relations

$$G^{\cdot}_N = -q_N, \qquad G^{\cdot}_P = -q_P \qquad (q_N(t),\ q_P(t) \geqslant 0) \tag{1.10}$$

After such a substitution the functions G_N and G_P are the phase coordinates, while q_N and q_P are the new controls (this method was previously used in [5]). If the weights G_N and G_P remain invariant along the trajectory under the conditions of the problem, then $q_N = 0$ and $q_P = 0$.

In order to solve the variational problem according to the method of L.S. Pontriagin, we form the Hamiltonian function H and write down the equations for the momenta

$$H = -p_\Sigma \frac{\alpha g}{2\gamma^2}\frac{G_P{}^2P^2}{G_N N} - p_N q_N - p_P q_P + \mathbf{p}_r \cdot \mathbf{v} + \mathbf{p}_v \cdot \left(\frac{g}{\gamma}\frac{PG_P}{G_\Sigma + G_N + G_P}\,\mathbf{i} + \mathbf{R}\right) \tag{1.11}$$

$$p^{\cdot}_\Sigma = \frac{g}{\gamma}\frac{PG_P}{(G_\Sigma + G_N + G_P)^2}(\mathbf{i}\cdot\mathbf{p}_v), \qquad \mathbf{p}^{\cdot}_r = -\frac{\partial}{\partial \mathbf{r}}(\mathbf{p}_v \cdot \mathbf{R})$$

$$p^{\cdot}_N = -p_\Sigma \frac{\alpha g}{2\gamma^2}\frac{G_P{}^2P^2}{G_N{}^2N} + \frac{g}{\gamma}\frac{PG_P}{(G_\Sigma + G_N + G_P)^2}(\mathbf{i}\cdot\mathbf{p}_v), \quad \mathbf{p}^{\cdot}_v = -\mathbf{p}_r \tag{1.12}$$

$$p^{\cdot}_P = p_\Sigma \frac{\alpha g}{2\gamma^2}\frac{G_P P^2}{G_N N} + \frac{g}{\gamma}\frac{P(G_\Sigma + G_N)}{(G_\Sigma + G_N + G_P)^2}(\mathbf{i}\cdot\mathbf{p}_v)$$

The boundary conditions for the momenta are

$$p_N{}^{(0)} = p_P{}^{(0)} = p_\Sigma{}^{(0)}, \qquad p_N{}^{(1)} = p_P{}^{(1)} = 0, \qquad p_\Sigma{}^{(1)} = -1 \tag{1.13}$$

From the condition of minimum H with respect to the control $\mathbf{i}$ it follows that

$$\mathbf{i} = -\mathbf{p}_v / p_v \tag{1.14}$$

The function $1/N$ enters into H linearly, hence the control N takes on the values 0 and 1 depending on the sign of the momentum $p_\Sigma(t)$.

We can show that $p_\Sigma(t) < 0$ over the entire interval $[0,T]$. If $p_\Sigma(t) > 0$ on some portion of the interval, then for this portion the optimum values of $P(t)$ and $N(t)$ will be the limiting values $P = 1$ and $N = 0$ (see (1.11)). This corresponds to infinite flow through the propulsion system with zero exhaust velocity. Such an operating condition is clearly not optimum, hence the assumption $p_\Sigma(t) > 0$ is untrue and consequently $p_\Sigma(t) < 0$ for $0 \leqslant t \leqslant T$.

In view of the constant sign of the function $p_\Sigma(t)$, the control function $N(t)$ is constant

$$N(t) = 1 \tag{1.15}$$

The control function $P(t)$ changes within the limits of the closed interval $1 \geqslant P(t) \geqslant 0$ as follows:

Within the interval

$$P(t) = P_{\rm opt} \qquad \left(P_{\rm opt} = -\frac{p_v}{p_\Sigma}\frac{\gamma}{\alpha}\frac{G_N N}{G_P(G_\Sigma + G_N + G_P)}\right) \tag{1.16}$$

At the upper limit

$$P(t) = 1 \qquad \text{for } P_{\rm opt} \geqslant 1 \tag{1.17}$$

At the lower limit

$$P = 0 \ \text{ for } \Delta > 0 \ \left(\Delta = -p_\Sigma \frac{\alpha g}{2\gamma^2}\frac{G_P{}^2P^2}{G_N N} - p_v\frac{g}{\gamma}\frac{PG_P}{G_\Sigma + G_N + G_P}\right) \tag{1.18}$$

With the help of Expression (1.16) for P_{opt} we transform the combination Δ into

$$\Delta = p_\Sigma \frac{\alpha g}{2\gamma^2} \frac{G_P^2 P^2}{G_N N} \left(-1 + \frac{2P_{opt}}{P}\right)$$

The ratio $P_{opt}/P \geqslant 1$, hence $\Delta < 0$ on the whole interval, and the lower limit (1.18) is not attained by the optimum control function $P(t)$ – the propulsion system is cut-in for the whole trajectory.

If limits of the type $q_P \leqslant Q_P$ and $q_N \leqslant Q_N$ are not imposed on the controls q_P and q_N, then the momenta p_P and p_N are nonpositive everywhere on $[0,T]$. We will carry out the proof for p_P. We assume the contrary, let $p_P(t') > 0$ at a certain instant $t = t'$. Then the control q_P takes on the optimum value $q_P' = \infty$ (see (1.11)), which corresponds to the instantaneous jettisoning of final part of the weight G_P. The derivative $\dot p_P$ remains finite for $q_P' = \infty$, hence the momentum $p_P(t)$ does not change sign in a finite interval of time in the neighborhood of the instant $t = t'$. Consequently, at a finite interval of time $q_P(t) = \infty$. This corresponds to an infinitely large jettisoning weight of G_P, which is impossible because of the finiteness of the component G_P. Hence the initial assumption is untrue and

$$p_P(t) \leqslant 0, \qquad p_N(t) \leqslant 0 \qquad (T \geqslant t \geqslant 0) \tag{1.19}$$

In the case, when conditions of the type mentioned are imposed on the control functions q_P and q_N and, in particular, it is assumed that $q_N = q_P = 0$, $(T \geqslant t \geqslant 0)$, then the conclusions regarding the signs of $p_N(t)$ and $p_P(t)$ are invalid.

The analysis of the optimum controls q_N and q_P which in accordance with the proposed method replace the old controls G_N and G_P gives two following types of regimes: regimes of limiting controls $q_N = 0$ and $q_P = 0$ for $p_N < 0$ and $p_P < 0$ which correspond to $G_N = \text{const}$ and $G_P = \text{const}$ and regimes of the singular controls $\dot p_N(t) = 0$ and $\dot p_P(t) = 0$ for $p_N(t) = 0$ and $p_P(t) = 0$ which correspond to a minimum of function H with respect to G_N and G_P. It should be noted that the presence of the two types of regimes mentioned is a general property of problems with boundary conditions of the type (1.9) imposed on the control functions.

The expressions for $\dot p_N$ and $\dot p_P$ with the aid of (1.16) may be transformed into

$$\begin{aligned}
\dot p_N &= p_v \frac{g}{\gamma} \frac{PG_P}{2G_N(G_\Sigma + G_N + G_P)} \left(\frac{P}{P_{opt}} - \frac{2G_N}{G_\Sigma + G_N + G_P}\right) \\
\dot p_P &= p_v \frac{g}{\gamma} \frac{P}{G_\Sigma + G_N + G_P} \left(-\frac{P}{P_{opt}} + \frac{G_\Sigma + G_N}{G_\Sigma + G_N + G_P}\right)
\end{aligned} \tag{1.20}$$

The regime of the singular control for q_N and q_P is realized respectively for

$$\frac{P}{P_{opt}} = \frac{2G_N}{G_\Sigma + G_N + G_P}, \qquad \frac{P}{P_{opt}} = \frac{G_\Sigma + G_N}{G_\Sigma + G_N + G_P} \tag{1.21}$$

If $P = P_{opt}$, then the second condition (1.21) obviously cannot be realized, and $\dot p_P < 0$. Consequently, the control P has its limiting value $P = 1$ on the portions of the trajectory on which G_P decreases.

In order to satisfy the boundary condition $p_P^{(1)} = 0$ in the case where there is no upper limit on $q_P(t)$, it is necessary that the trajectory be completed by a portion with $P \neq 1$. Actually, the momentum $p_P(t)$ is nonpositive at every instant of time (1.19), hence in order to attain the upper limit $p_P = 0$ the derivative $\dot p_P$ must be nonnegative to the left of the point $p_P = 0$, which according to (1.20) can happen only for $P = 1$.

To conclude this Section, we cite the integral [4] of the systems (1.8) and (1.12), which exists for $P = P_{opt}$, $G_N = \text{const}$ and $G_P = \text{const}$

$$(G_\Sigma + G_N + G_P)^2 p_\Sigma = \text{const}$$

2. We consider the variational problem for an ideal propulsion system, taking into account the weight of the reactive mass of the power source $G = G_n + G_m + G_N + G_\varepsilon$.

The weight of the reactive mass G_ε is expressed in terms of the energy E delivered as the mass as

$$G_\varepsilon = Eg / c^2\eta \qquad (\eta \approx 5 \cdot 10^{-4})[6] \tag{2.1}$$

where c is the velocity of light and η is the coefficient of transformation of mass into energy.

The power and the available energy at the instant t are related by

$$E^{\cdot} = -N \tag{2.2}$$

We introduce the weight sum $G_s = G_n + G_m + G_\varepsilon$.

As in Section 1, we refer the power to the maximum power $N_{\max} = G_N / \alpha$ and the weights G_s, G_n and G_N to the initial weight $G^{(0)}$, retaining the old notation for the new dimensionless quantities. The differential equations for consumption of weight G_s and for the dynamics, as well as the initial condition for the weight G_s, become

$$G_s^{\cdot} = -\frac{g}{c^2\eta\alpha} NG_N - \frac{\alpha}{2g}\frac{(G_s + G_N)^2}{NG_N} a^2, \qquad \mathbf{r}^{\cdot} = \mathbf{v} \tag{2.3}$$

$$\mathbf{v}^{\cdot} = a\mathbf{i} + \mathbf{R}; \qquad G_s^{(0)} + G_N^{(0)} = 1$$

The variational problem consists of determining the optimum controls N, G_N, $\mathbf{i}$ and a which yield a maximum of the functional $G_s^{(1)} = G_n$.

Constructing the Hamiltonian function and writing the equations for the momenta, we have

$$H = -p_s\left[\frac{g}{c^2\eta\alpha} NG_N + \frac{\alpha}{2g}\frac{(G_s + G_N)^2}{NG_N} a^2\right] + \mathbf{p}_r\cdot\mathbf{v} + \mathbf{p}_v\cdot(a\mathbf{i} + \mathbf{R}) \tag{2.4}$$

$$p_s^{\cdot} = p_s \frac{\alpha}{g}\frac{G_s + G_N}{NG_N} a^2, \qquad \mathbf{p}_r^{\cdot} = -\frac{\partial}{\partial \mathbf{r}}(\mathbf{p}_v\cdot\mathbf{R}), \qquad \mathbf{p}_v^{\cdot} = -\mathbf{p}_r \tag{2.5}$$

$$p_s(T) = -1$$

The control $\mathbf{i}(t)$ yielding the minimum H is given by Formula (1.12). The method developed in Section 1 should be applied in order to determine the optimum control $G_N(t)$. Here we assume that $G_N = \text{const}$.

We write out the part of the function H containing $a(t)$ and $N(t)$

$$H^* = -p_s\left[\frac{g}{c^2\eta\alpha} NG_N + \frac{\alpha}{2g}\frac{(G_s + G_N)^2}{NG_N} a^2\right] - p_v a \tag{2.6}$$

The momentum p_s is negative everywhere in the interval $0 \leqslant t \leqslant T$. Actually, the function $p_s(t)$ is determined by the boundary condition and the homogeneous differential equation (2.5) with bounded coefficients, hence the function $p_s(t)$ does not change sign and remains negative everywhere.

The minimum of the function H^* with respect to $a(t)$ and $N(t)$ is attained under conditions

$$a = -\frac{p_v}{p_s}\frac{g}{\alpha}\frac{G_N}{(G_s + G_N)^2}, \qquad N = 1 \quad \text{for } \Delta_\varepsilon < 0$$

$$a = 0, \qquad N = 0 \quad \text{for } \Delta_\varepsilon > 0 \tag{2.7}$$

$$\Delta_\varepsilon = -p_s \frac{g}{c^2\eta\alpha} G_N + \frac{p_v^2}{p_s}\frac{g}{2\alpha}\frac{G_N}{(G_s + G_N)^2}$$

Thus the consideration of the weight of the reactive mass for the power source leads to the possible inclusion of passive intervals as parts of the optimum trajectory. This occurs when the acceleration, calculated from

(2.7), satisfies the inequality

$$a < g \frac{\sqrt{2/\eta}}{c\alpha} \frac{G_N}{G_s + G_N}, \quad \text{or} \quad V > c\sqrt{2\eta}$$

the latter is expressed in the terms of the exhaust velocity.

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